Maximizing the Guarded Boundary of an Art Gallery is APX-complete [⋆]

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Abstract

In the Art Gallery problem, given is a polygonal gallery and the goal is to guard the gallery's interior or walls with a number of guards that must be placed strategically in the interior, on walls or on corners of the gallery. Here we consider a more realistic version: exhibits now have *size* and may have different costs. Moreover the meaning of guarding is relaxed: we use a new concept, that of *watching* an expensive art item, i.e. overseeing a *part* of the item. The main result of the paper is that the problem of maximizing the total value of a guarded weighted boundary is APX-complete. This is shown by an appropriate 'gap-preserving' reduction from the Max-5-occurrence-3-Sat problem. We also show that this technique can be applied to a number of maximization variations of the art gallery problem. In particular we consider the following problems: given a polygon with or without holes and k available guards, maximize a) the *length of walls* guarded and b) the *total cost* of paintings *watched* or *overseen*. We prove that all the above problems are APX-complete.

Key words: Approximation Algorithms, Art Gallery, Visibility.

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1. Introduction

In the Art Gallery problem (as posed by Victor Klee during a conference in 1976), we are asked to place a minimum number of guards in an art gallery so that every point in the interior of the gallery can be seen by at least one guard.

Besides its application of guarding exhibits in a gallery, the Art Gallery problem has applications in wireless communication technology (mobile phones, etc): place a minimum number of stations in a polygonal area so that any point of the area can communicate with at least one station (two points can communicate if they are mutually visible).

Many variations of the Art Gallery problem have been studied during the last two decades [20–22]. These variations can be classified with respect to where the guards are allowed to be placed (e.g. on vertices, edges, interior of the polygon) or whether only the boundary or all of the interior of the polygon needs to be guarded, etc. Most known variations of this problem are NP-hard [13]. Related problems that have been studied are Minimum Vertex/Edge/Point Guard for guarding the boundary or area of a polygon with or without holes (APX-hard [7,8] and O($\log n$)-approximable for vertex or edge guards [11]) and Minimum Fixed Height Vertex/Point Guard On Terrain ($\Theta(\log n)$-approximable [7,8]). Furthermore, it has been proved that the Minimum Point Guard problem remains APX-hard for some special polygon families: 3-link polygons ([17]) and 2-link polygons ([3]). In [18] they give a constant factor approximation algorithm for interior guarding of monotone polygons. They also give an approximation algorithm for interior guarding rectilinear polygons that produces a guard set of size $O(OPT^2)$. It is not known whether the Minimum Point Guard problem in arbitrary polygons can be approximated within a $O(\log n)$ factor.

In [12] the case of guarding the walls (and not necessarily every interior point) is studied. In [6] the following problem has been introduced: suppose we have a number of valuable treasures in a polygon P; what is the minimum number of mobile (edge) guards required to patrol P in such a way that each treasure is always visible from at least one guard? In [6] they show NP-hardness and give heuristics for this problem. In [4] weights are assigned to the treasures in the gallery. They study the case of placing one guard in the gallery in such a way that the sum of weights of the visible treasures is maximized. In [19] they study the case of placing one guard in the polygon (anywhere inside) so that a maximum area of the polygon is covered. They give an algorithm which achieves $1-\delta$ approximation of the maximum area, for any $\delta > 0$ (PTAS). The time complexity of their algorithm is $O(n^5/\delta^2)$ [3]. The problem of placing a given number k of guards inside the polygon so that a maximum area of the polygon is covered has been studied in [5]. They give an algorithm for this problem which achieves with high probability, $1-e^{\delta-1}$ approximation of the maximum area for $\delta > 0$ (i.e. infinitely close to $1-\frac{1}{e}$), in time $O((k^3n^2/\delta^4)\log^3(n/\delta))$.

In this paper we first focus on the last problem with the following restrictions: a) allowing guards to be located only on the boundary and b) restricting the intended guarded area to be on the boundary of the polygon. We prove that with the first restriction the problem is APX-hard no matter whether the intended guarded area is the boundary

[3] Actually it is a Fully Polynomial Time Approximation Scheme (FPTAS) since the time complexity is polynomially related to n, $\frac{1}{\delta}$.

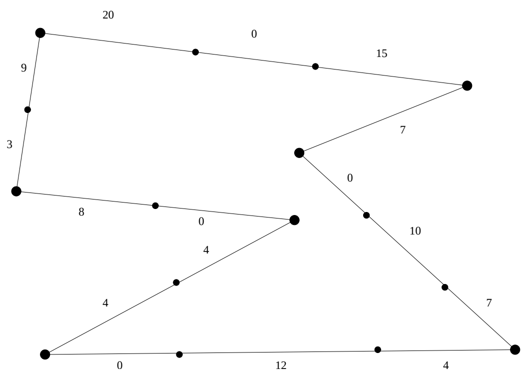

Fig. 1. A weighted polygon.

or the interior of the polygon. We get this result by introducing and studying another related art gallery problem (which has its own applications).

More specifically we consider here the Maximum Length Vertex Guard problem: A polygon without holes and an integer $k > 0$ are given; The goal is to place at most k guards on vertices of the polygon so that a maximum part of the boundary is visible by the guards. We prove that this problem is APX-hard. To get this result, we define and prove APX-hardness of another related problem called Maximum Value Vertex Guard problem in which the boundary of the polygon consists of weighted disjoint line segments (see Figure 1). The goal in the latter problem is to place at most k guards on vertices of the polygon so that the total weight of line segments visible by the guards is maximized. This latter problem has its own applications: If we think of the weighted line segments as paintings on the walls of an art gallery then we have a realistic abstraction of the problem of guarding a maximum total value of paintings that takes into account the fact that paintings actually occupy parts of the walls, not merely points. Another possible application of this problem is the illumination of a maximum number of paintings in a gallery. Again, a painting must be totally visible from light sources in order to consider it illuminated. There are also important applications in wireless communication networks: An interpretation of weighted line segments are inhabited areas. The polygon models the geographical space. The weight interpretation is the population of an area. Imagine a number of towns lying on the boundary of a polygonal geographical area. The goal is to place at most k stations such that the total number of people that can communicate is maximized. Moreover, it could be the case that the towns are on the shore of a lake, so we can only place stations on the boundary. Similar situations may arise in various other types of landscape.

We show APX-hardness of Maximum Value Vertex Guard and Maximum Length Vertex Guard and conclude that these problems are APX-complete since there exist polynomial time constant-ratio approximation algorithms ([15]). Our main contribution is a 'gap-preserving' reduction from Max-5-occurrence-3-Sat to Maximum Value Vertex Guard specially designed for weighted maximization problems which also applies for the Maximum Length Vertex Guard problem. The construction part of our reduction uses some ideas from the constructions used in [13], [8] (to show NP-hardness, APX-hardness respectively of the Minimum Vertex Guard problem). Central in our technique is a careful assignment of appropriate weights on the line segments of the

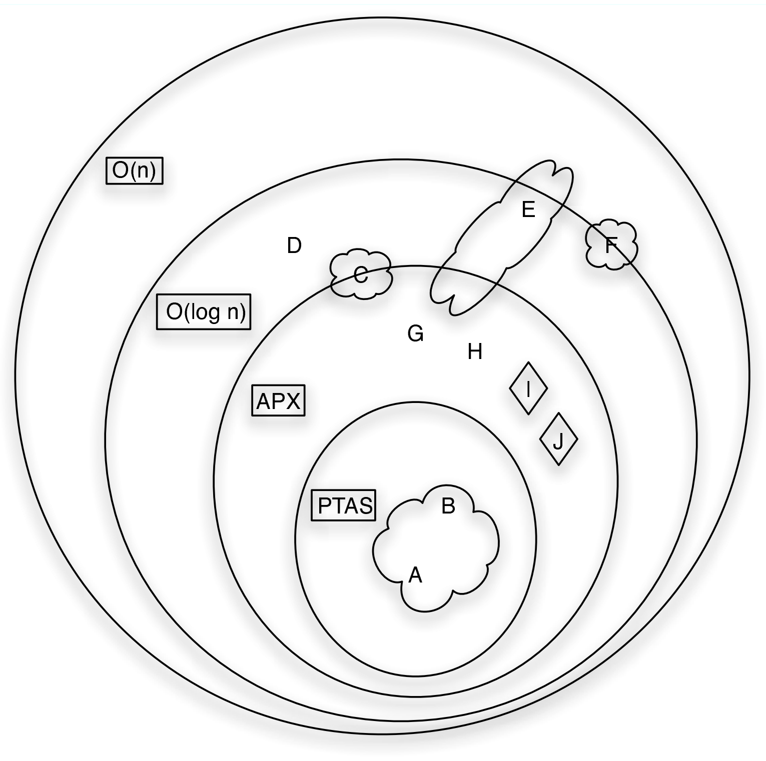

Fig. 2. Classifying Art Gallery problems in approximation classes. We use: "$O(n)$" to denote the class of problems with $O(n)$ approximation ratio, "$O(\log n)$" for the class of problems with $O(\log n)$ approximation ratio, APX for the class of problems with constant approximation ratio and PTAS for the class of problems with an infinitely close to 1 constant approximation ratio.

constructed polygon.

Next we study a couple of variations: a) the case of edge guards (guards occupying whole edges), and b) the case in which our goal is to *watch* (see a part of) line segments instead of overseeing them. We also prove APX-completeness for those variations. In Figure 2 we can see a picture of the hierarchy of some art gallery problems. The problems in Figure 2 are denoted as shown in the first column of the table in Figure 3. They have been placed in the hierarchy by researchers as shown in the last column of the table in Figure 3.

We also use in Figure 2 the 'cloud' notation to indicate that a problem's position is somewhere in the cloud: problem C has been proved APX-hard ([7,8]) but only $O(\log n)$ approximation algorithms are known ([11]); problem E has been proved APX-hard but only $O(n)$ approximation algorithms are known; problem F has been proved $\log n-$hard but only $O(n)$ approximation algorithms are known; problems A, B have $PTAS$ schemas ([19]) but it has not been proved whether these problems are $NP-hard$; problems I, J are enclosed in a 'diamond' notation to indicate that the constant-ratio approximation algorithms for them are probabilistic ones ([5]); finally, problems D, G, H belong exactly to the classes figured (D is $\log n-$hard and $O(\log n)$ approximable, while G, H are APX-complete).

A	One Point Guard without holes	[19]
B	One Point Guard with holes	[19]
C	Minimum Vertex/Edge Guard without holes	[11,7,8]
D	Minimum Vertex/Edge Guard with holes	[11,7,8]
E	Minimum Point Guard without holes	[7,8]
F	Minimum Point Guard with holes	[7,8]
G	Maximum Length/Area Vertex/Edge Guard without holes	here, [10]
H	Maximum Length/Area Vertex/Edge Guard with holes	here, [10]
I	Maximum Length/Area Point Guard without holes	[5]
J	Maximum Length/Area Point Guard with holes	[5]

Fig. 3. Classification of Art Gallery problems in approximation classes.

2. Guarding a maximum part of a polygon's boundary is APX-complete

Suppose a polygon P without holes is given. The goal is to place k vertex guards so that the overseen part of the boundary is maximized. The formal definition follows:

Definition 1 *Given is a polygon P without holes and an integer $k > 0$. Let $L(b)$ be the Euclidean length of the line segment b. The* Maximum Length Vertex Guard *problem asks to place k vertex guards so that the Euclidean length of the* overseen *part of P's boundary is maximum.*

We will prove the APX-hardness of that problem by studying another related problem (which has also its own importance): Suppose there are weighted disjoint line segments on P's boundary. We denote these segments as open intervals (a, b). The goal is to place k vertex guards maximizing the weight of the overseen boundary. The formal definition follows:

Definition 2 *Given is a polygon P without holes and an integer $k > 0$. Assume the boundary of P is subdivided into disjoint line segments with non negative weights (see Figure 1). The goal of the* Maximum Value Vertex Guard *problem is to place k vertex guards so that the total weight of the set of line segments overseen is maximum.*

We will prove that Maximum Value Vertex Guard is APX-hard. We propose a reduction from Max-5-occurrence-3-Sat problem (known to be APX-hard [2]) and we show that it is a 'gap preserving reduction' [4] . Let us recall the formal definition of the Max-5-occurrence-3-Sat problem:

Definition 3 *Let Φ be a boolean formula given in conjuctive normal form, with each clause consisting of at most 3 literals and with each variable appearing in at most 5 clauses. The goal of* Max-5-occurrence-3-Sat *problem is to find a truth assignment for the variables of Φ such that the number of satisfied clauses is maximum.*

[4] A 'gap-preserving' reduction [2] from a problem A to a problem B is a polynomial time algorithm which transforms some instances of A (namely instances for which there is a 'gap' between their optimal solutions) to instances of B preserving the 'gap' notion.

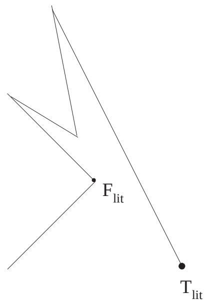

Fig. 4. A literal pattern.

Fig. 5. A clause pattern.

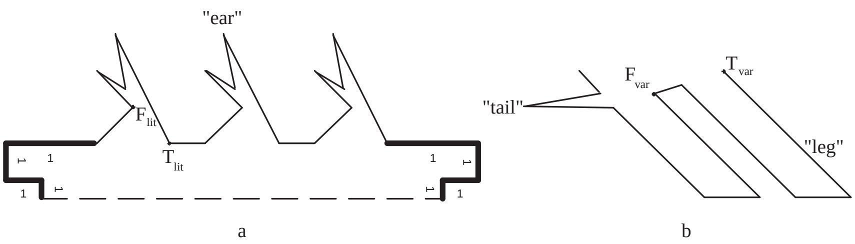

Fig. 6. a) A clause pattern with 3 literal patterns, b) a variable pattern.

2.1. *Construction part of the reduction*

In order to construct an instance of our problem, we will augment the construction proposed in [8] for Minimum Vertex Guard problem, by assigning appropriate weights. Let us recall the construction proposed in [8]:

For every literal of the given boolean expression, we construct a literal pattern as shown in Figure 4. The literal pattern has been constructed in such a way that the following holds: it is possible to oversee the whole literal pattern with one vertex guard *only* if she is placed on vertex F_{lit} or T_{lit}.

For every clause of the given boolean expression, we construct a clause pattern as shown in Figure 5. In order to oversee this clause pattern with exactly one guard, this guard has to be placed inside the area which is shown in Figure 5 (or on the boundary). Figure 6a shows a clause pattern with 3 literal "ear" patterns.

For every variable of the given boolean expression, we construct a variable pattern as shown in Figure 6b. We call 'legs' the two bottom parts of the pattern and 'tail' the left part. Variable patterns are augmented with additional spikes described below (see Figure 7).

Finally we add an "ear" pattern in the upper left corner of the polygon and the construction is complete. A guard on vertex w oversees both "legs" of every variable pattern. An example is shown in Figure 8.

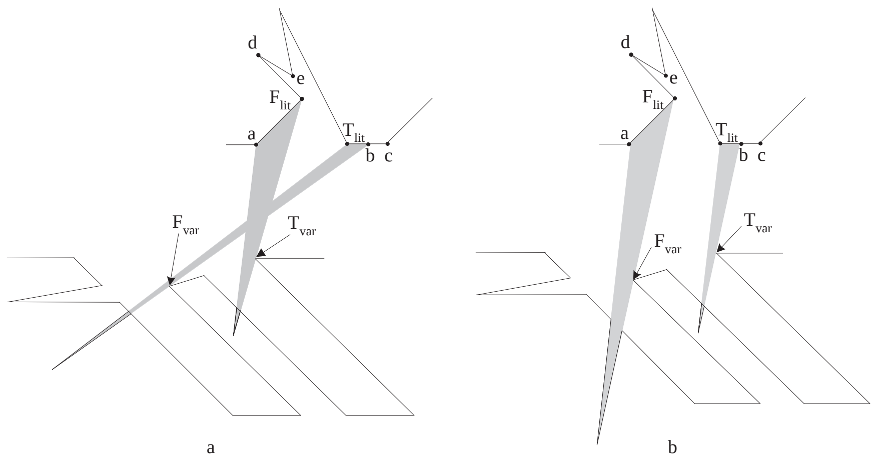

Fig. 7. a) Two spikes corresponding to an occurrence of a positive literal in a clause. Both spikes and the "ear" are overseen by two guards placed, e.g., on F_{var} (oversees left spike and "tail") and on F_{lit} (oversees right spike and "ear"). b) Two spikes corresponding to an occurrence of a negative literal in a clause. Both spikes and the "ear" are overseen by two guards placed, e.g., on T_{var} (oversees right spike and "tail") and on F_{lit} (oversees left spike and "ear").

For every occurrence of a literal in the boolean expression, i.e. for every literal "ear" pattern, we add two "spikes" to the corresponding variable pattern: if it is a positive (negative) literal, we add the two spikes as shown in Figure 7a (7b). The spike which is overseen by vertex F_{lit} (T_{lit}) is called FALSE (TRUE) spike. Notice in Figure 7 that the base of the FALSE spike is the line segment (a, F_{lit}), whereas the base of the TRUE spike is (T_{lit}, b) and not (T_{lit}, c). The purpose of this is that no vertex of the clause side (see Figure 7) can oversee more than one spike (in the variable side). The orientation of the legs of the variable patterns is such that they can be overseen by one guard placed on vertex w (see Figure 8).

As observed in [8], three guards are necessary and sufficient in order to oversee a literal "ear", its corresponding variable pattern (two "legs" and a "tail") and its corresponding spikes. One of them is placed on vertex w and oversees the "legs" of the variable pattern. The other two are placed on vertices: i) $\{F_{var}, F_{lit}\}$, or $\{T_{var}, T_{lit}\}$, for positive literals, or ii) $\{F_{var}, T_{lit}\}$, or $\{T_{var}, F_{lit}\}$, for negative literals.

Now we will augment the previous construction as follows: We assign value 8 to every edge of the polygon, except the "cheap" edges of the clause patterns depicted in Figure 6a, to which we assign value 1. We set the number of available guards $k = l + n + 1$, where l is the number of occurrences of literals and n is the number of variables of the boolean expression.

2.2. *Transformation of a feasible solution*

Suppose a truth assignment for the boolean expression is given. We will construct a guard placement that corresponds to the given truth assignment. We place $k = l + n + 1$ guards on vertices of the polygon that we constructed in section 2.1, as follows: We place in each variable pattern a guard on vertex F_{var} (T_{var}), if the truth value of the

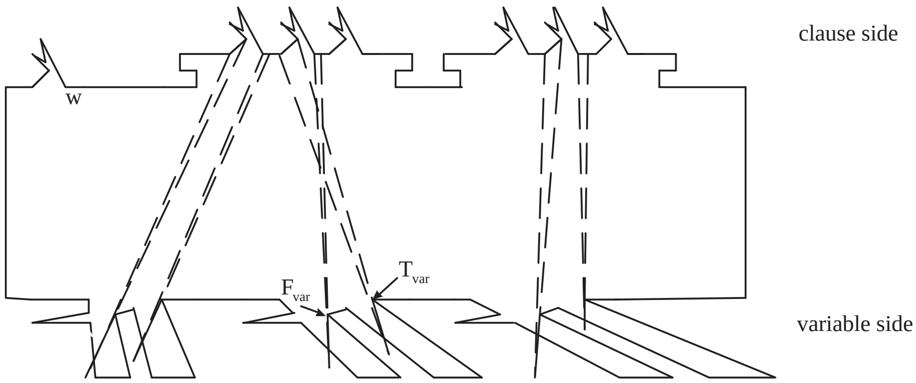

Fig. 8. Resulting polygon.

corresponding variable is FALSE (TRUE). We place in each literal pattern a guard on vertex F_{lit} (T_{lit}), if the truth evaluation of the literal is FALSE (TRUE). Finally we place a guard in the additional "ear" pattern, on vertex w. Thus, every literal pattern is overseen. Furthermore, every variable pattern (together with its spikes) is overseen by guards placed as described. The "legs" of variable patterns are overseen by the guard on vertex w. The only edges of the constructed polygon possibly not overseen are the edges of the clause patterns corresponding to unsatisfiable clauses.

Conversely, given a placement of $l + n + 1$ guards on the resulting polygon which is an instance of Maximum Value Vertex Guard we will construct a corresponding truth assignment for the original Max-5-occurrence-3-Sat instance. First we modify the placement of guards by placing a) only one guard in every variable pattern on one of the vertices F_{var} or T_{var}, b) only one guard in every literal pattern on vertex F_{lit} (T_{lit}) if the corresponding TRUE (FALSE) spike of the variable pattern is overseen by its guard, c) one guard in the additional "ear" pattern on vertex w. In more details: given a placement of $k = l + n + 1$ guards with a total overseen boundary value B, we will modify the guard placement so that the total value overseen is $\geq B$, and so that with the exception of some "cheap" edges with weight 1, the modified guard placement achieves: a) full overseeing of all polygon edges and b) "consistent" placement on two vertices out of the four $F_{lit}, T_{lit}, F_{var}, T_{var}$ for all literals. Guard placement follows:

i) We place one guard on vertex w of the additional "ear" pattern.

ii) For every variable pattern: a) If there is only one guard in the pattern placed on a vertex which oversees a spike, we place her on F_{var} (T_{var}) if F_{var} (T_{var}) oversees the same spike. b) In all other cases (no guards, one guard overseeing no spikes, at least two guards) we place one guard on F_{var} (T_{var}) if F_{var} (T_{var}) oversees more FALSE spikes than T_{var} (F_{var}).

iii) For every literal we place one guard on F_{lit} (T_{lit}) if the corresponding FALSE (TRUE) spike of the variable pattern is not overseen by the guard placed in the variable pattern.

We will prove in section 2.3 (see Lemma 2.2) that the total value overseen is at least B.

Now we can construct a truth assignment as follows: assign TRUE (FALSE) to a variable if the corresponding variable pattern has a guard on vertex T_{var} (F_{var}).

2.3. *Analysis of the reduction*

Let I be an instance of MAX-5-OCCURRENCE-3-SAT with n variables, l occurrences of literals and m clauses ($l \leq 3m$). Let I' be the instance of MAXIMUM VALUE VERTEX GUARD (constructed as in section 2.1) with $k = l + n + 1$. Let M be the total value of the boundary.

Lemma 2.1 *If* $OPT(I) = m$ *then* $OPT(I') = M$.

Proof: Suppose there exists a truth assignment such that all m clauses are satisfied. If we place $l + n + 1$ guards in the polygon as in section 2.2, then it is easy to see that the whole boundary of the polygon is overseen. So the total value of overseen edges is M. □

Note that Lemma 2.1 is true no matter what the values of the cheap edges are. However we must carefully choose the values of cheap edges in order to prove Lemma 2.2. We want to find an optimal placement of guards in which for many clause patterns at least one of the T_{lit} vertices is occupied by a guard. Thus the values of cheap edges should not be 0. We also want to cover all non-cheap edges possibly leaving some cheap ones uncovered. For every false clause of the boolean formula cheap edges in the corresponding clause pattern will be left uncovered.

Lemma 2.2 *If* $OPT(I') \geq M - 8\epsilon m$ *then* $OPT(I) \geq m(1 - \epsilon)$.

Proof: Suppose there exists an $\epsilon > 0$ and a placement of the $l+n+1$ guards in I' so that the total value of overseen boundary is at least $M - 8\epsilon m$. After the modification of guard placement described in section 2.2, $k = l + n + 1$ guards oversee the whole boundary except possibly some "cheap" edges. We prove that the total value overseen is again at least $M - 8\epsilon m$.

Notice that if we place a guard on vertex F_{lit} or T_{lit} of an "ear" (belonging to a literal pattern) which has no guards we certainly increase the overseen value by at least 16, since edges (F_{lit}, d) and (d, e) could not be overseen by a guard place outside of the literal pattern. Similarly a guard placed on F_{var} or T_{var} of a variable pattern that has no guards, certainly increases the overseen value by at least 16 (namely weight of the two "tail" edges).

We will discuss two cases pertaining to guard placement in "ears":

a) Suppose that in the original placement, two guards had been placed on both vertices T_{lit} and F_{lit} of a literal "ear" pattern and after the modification, there is only one guard placed inside the literal pattern on vertex F_{lit}. The total value may have been decreased by at most 8 (because "cheap" edges may now be missed) but it is increased by at least 16 (because the free guard was placed in an unguarded pattern).

b) Suppose that in the original placement exactly one guard had been placed inside a literal "ear" pattern on vertex T_{lit} and after the modification she was moved to vertex F_{lit} of the pattern:

i) If the corresponding FALSE spike was not overseen in the original placement (by a guard in the variable pattern), the total value may have been decreased by at most 8 (because "cheap" edges may now be missed) but it is increased by at least 16 (because the FALSE spike is now overseen by the guard on F_{lit}).

ii) If the corresponding FALSE spike was overseen in the original placement (by a guard g in the variable pattern), then there were at least two guards placed inside the variable pattern initially, since otherwise (if there was only one guard initially), the guard would oversee the FALSE spike and according to step 3 of the modification procedure as described in section 2.2, a guard would have been placed to T_{lit} and not F_{lit}. Therefore it must be the case that the variable pattern had originally at least two guards and after the modification, guard g was removed and placed in another pattern because there was another guard that was overseeing the most FALSE spikes in the variable pattern. This means that the guard g was overseeing at most 2 FALSE spikes because a variable pattern has at most 5 FALSE spikes, since a variable appears in at most 5 clauses of the boolean formula. Thus, for every variable pattern, guards have been moved from vertex T_{lit} to F_{lit} in at most two literal patterns. The total value may have been decreased by at most 16 (because "cheap" edges of two clauses may now be missed) but it is increased by at least 16 (because at least one free guard was placed in an unguarded pattern).

After modification, the guards oversee the boundary of the polygon except possibly at most ϵm clause patterns. We can now construct a truth assignment for I as in section 2.2 that leaves at most ϵm clauses unsatisfied that correspond to the ϵm clause patterns not overseen by any guard in I'. □

From Lemma 2.1 and the contraposition of Lemma 2.2 the following proposition holds:

Proposition 4 *Let I be an instance of* MAX-5-OCCURRENCE-3-SAT *problem with n variables, m clauses and $l \leq 3m$ occurrences of literals. Let I' be the instance of* MAXIMUM VALUE VERTEX GUARD *problem (constructed as in section 2.1) with $k = l + n + 1$. Let M be the total value of the boundary of the polygon. Then:*

– $OPT(I) = m \rightarrow OPT(I') = M$

– $OPT(I) \leq m(1-\epsilon) \rightarrow OPT(I') \leq M - 8\epsilon m$

In [1,2] it was proved that given an instance I of the MAX-5-OCCURRENCE-3-SAT problem for which either $OPT(I) = m$ or $OPT(I) \leq m(1-\epsilon)$ for some $\epsilon > 0$ (where m denotes the number of clauses), it is NP-hard to decide which one of the above holds. Therefore, we obtain that unless $P = NP$, no polynomial time approximation algorithm for MAXIMUM VALUE VERTEX GUARD can achieve an approximation ratio between OPT and SOL (where SOL is the approximation solution returned) smaller than $\frac{M}{M-8\epsilon m}$, since if such an algorithm exists it can be used to decide in polynomial time whether $OPT(I) = m$ or $OPT(I) \leq m(1-\epsilon)$. To see why, suppose for a moment that such an approximation algorithm exists and returns a solution $SOL(I')$, where $\frac{OPT(I')}{SOL(I')} < \frac{M}{M-8\epsilon m}$: if $SOL(I') > M - 8\epsilon m$, then $OPT(I') > M - 8\epsilon m \rightarrow OPT(I) > m(1-\epsilon) \rightarrow OPT(I) = m$; if $SOL(I') \leq M - 8\epsilon m$, then (using the small approximation ratio) $OPT(I') < M \rightarrow OPT(I) < m \rightarrow OPT(I) \leq m(1-\epsilon)$.

Considered that $M = nV + lL + 2lS + mC + E$ where V denotes the total value of a variable pattern ("legs", "tail", "leg-edges" between spikes, plus one edge that links the variable pattern with the next one on the right : $104 \leq V \leq 168$), L denotes the total value of a literal pattern ("ear" : $L = 40$), S denotes the total value of a spike pattern ($S = 16$), C denotes the total value of a clause pattern without "ears" plus one edge that links the clause pattern with the next one on the right ($16 \leq C \leq 32$) and E denotes the total value of the additional ear pattern and the remaining edges of the polygon ($E = 80$), then:

$$M \leq 3mV + 3mL + 6mS + mC + E$$

With a few calculations it turns out:

$$\frac{M}{M - 8\epsilon m} = \frac{1}{1 - \frac{8\epsilon m}{M}} \geq \frac{1}{1 - \frac{8\epsilon}{3V+3L+6S+C+E}} \geq 1 + \epsilon'$$

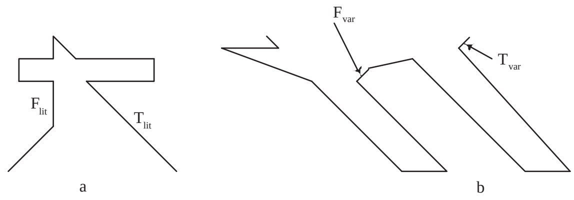

Fig. 9. a) A literal pattern and b) a variable pattern for edge guard problems.

for some ϵ' that depends on ϵ. Therefore, unless $P = NP$, no polynomial time approximation algorithm for Maximum Value Vertex Guard can achieve an approximation ratio smaller than $1 + \epsilon'$.

Theorem 5 Maximum Value Vertex Guard *is APX-hard.*

We now focus on the Maximum Length Vertex Guard problem.

Proposition 6 Maximum Length Vertex Guard *is APX-hard.*

Proof: For the construction part of the reduction, we construct the polygon as before with the following additional modification: we make sure that the length of every (previously) "cheap" edge in a clause pattern is designed at least 8 times shorter than any other edge of the polygon. If we replace the notion of value with that of length then the properties of Proposition 4 hold here as well. □

Our proof also works for the Maximum Area Vertex Guard problem. To see this, note that given a solution of a Max-5-occurrence-3-Sat instance for which all clauses are satisfied, if we place the guards in the transformed polygon as in Section 2.2, they oversee the whole boundary and area of the polygon. Furthermore, given a guards' placement on the transformed polygon for which the whole boundary and area of the polygon is overseen, then we can move the guards as in section 2.2 to get a solution for the Max-5-occurrence-3-Sat problem satisfying all clauses.

It has been proved in [15,10] that the problems Maximum Value Vertex Guard and Maximum Length/Area Vertex Guard can be approximated within a constant. Therefore:

Corollary 7 *The problems* Maximum Value Vertex Guard *and* Maximum Length/Area Vertex Guard *are APX-complete.*

3. Several APX-complete Art Gallery problems

In this section we propose appropriate modifications of the reduction of section 2 in order to show APX-hardness for a couple of variations of the previously discussed problems.

First we discuss the case in which guards are placed on edges (guards occupying whole edges). A guard which is occupying a whole edge, can be thought of as a mobile guard able to move on the edge.

Proposition 8 *The problems* Maximum Value Edge Guard *and* Maximum Length/Area Edge Guard *are APX-hard.*

The above result can be shown by a reduction from Max-5-occurrence-3-Sat to Maximum Value Edge Guard and Maximum Length/Area Edge Guard. The reduction follows the one in section 2 using modified literal and variable patterns, pro-

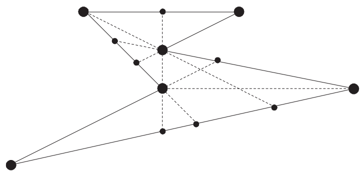

Fig. 10. Subdividing the boundary into line segments with endpoints in FVS.

posed in [8] for the Minimum Edge Guard problem. They are shown in Figure 9. It is not hard to check that the properties mentioned in Proposition 4 hold here as well.

We now discuss another variation of the Maximum Value Vertex/Edge Guard problem: we relax the meaning of guarding: "watching a valuable painting", i.e. "overseeing a part of it" instead of "overseeing all of it".

Let us describe a reduction from Max-5-occurrence-3-Sat to the watching version of Maximum Value Vertex/Edge Guard. We first construct the polygon using the appropriate gadgets (depending on the kind of guards as explained above). We then discretize the boundary using the Finest Visibility Segmentation (FVS) described in [15]. Let us recall this technique: we use the visibility graph $V_G(P)$. By extending edges of $V_G(P)$ inside P up to the boundary of P we obtain a set FVS of points on the boundary of P (FVS includes of course all corners of P) (see Figure 10). There are $O(|V(P)|^2)$ points in FVS and these points are endpoints of line segments with the following property: for any vertex v of the polygon, a segment (a, b) defined by consecutive FVS points is visible by v iff it is watched by v.

Let $\delta > 0$ be an integer such that the number of FVS segments in any of the (previously) "cheap" edges of a clause pattern is at most δ. We assign value 1 to every FVS segment which belongs to a (previously) "cheap" edge of a clause pattern. We assign value 8δ to every other segment.

We can prove similarly as before the following:

Proposition 9 *Let I be an instance of* Max-5-occurrence-3-Sat *problem with n variables, m clauses and $l \leq 3m$ occurrences of literals. Let I' be the instance of the watching version of* Maximum Value Vertex/Edge Guard *problem (constructed as in section 2.1) with $k = l+n+1$. Let M be the total value of the boundary of the polygon. Then:*
- $OPT(I) = m \rightarrow OPT(I') = M$
- $OPT(I) \leq m(1-\epsilon) \rightarrow OPT(I') \leq M - 8\delta\epsilon m$

We can argue as in the previous section that $\frac{M}{M-8\delta\epsilon m} \geq 1 + \epsilon''$ for some $\epsilon'' \geq 0$. Therefore:

Proposition 10 *The watching version of* Maximum Value Vertex/Edge Guard *problem is APX-hard.*

It is straightforward for one to see that all these problems are no easier in polygons with holes. Holes in polygons are useful because they give us the chance to model reality better (holes represent obstacles) and to place guards in the interior of the polygon (on predefined places), on vertices or edges of the holes. We remind the reader that for Minimum Vertex/Edge Guard for polygons with holes, no polynomial time approximation algorithm can guarantee an approximation ratio of $\frac{1-\epsilon}{12} \ln n$ for any $\epsilon > 0$, unless $NP \subseteq TIME(n^{O(\log \log n)})$ ([9,7]).

For all the above the problems, polynomial time constant ratio approximation algorithms are presented in [15]. Hence:

Theorem 11 *The problems* Maximum Length/Area Vertex/Edge Guard *and* Maximum Value Vertex/Edge Guard *(overseeing and watching version) are all APX-complete for polygons with or without holes.*

4. Conclusions

We have proved that overseeing a maximum part of the boundary or area of a polygon, using at most k vertex or edge guards, as well as several variations, are APX-complete. We hence contribute to the approximation hierarchy of the Art Gallery problems (Figure 2). It appears that some maximization art gallery problems are easier and some other are not harder than the minimization ones. An interesting question is whether the maximization problems remain APX-hard even for restricted polygon families (e.g. 2-link, 3-link polygons, etc).

We have shown that our 'gap-preserving' reduction can be applied with minor modifications to a number of problems. New elements of problems studied here are: a) **weighted line segments** of the polygon's boundary and b) the useful and promising concept of **watching** line segments as opposed to completely **overseeing** them. Interesting variations of these problems could follow if the potential guard positions have costs and there is a budget restriction (instead of a number of guards) [16]. A number of variations are also studied in [14].

Apart from the remaining open problems shown in Figure 2, interesting open problems arise if we consider the case where exhibits with costs may lie in the interior of the polygon.

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References

[1] S. Arora, Probabilistic Checking of Proofs and the Hardness of Approximation Problems, Ph.D. Thesis, UC Berkeley, 1994.

[2] S. Arora and C. Lund, Hardness of Approximations; in: D. Hochbaum, ed., Approximation Algorithms for NP-Hard Problems (PWS Publishing Company, Boston, 1996) 399-446.

[3] B. Broden, M. Hammar, B.J. Nilson, Guarding Lines and 2-Link Polygons is APX-hard, in: Proc. 13th Canadian Conf. on Computational Geometry, CCCG'01, 45-48, 2001.

[4] S. Carlsson and H. Jonsson, Guarding a Treasury, in: Proc. 5th Canadian Conf. on Computational Geometry, 85-90, 1993.

[5] O. Cheong, A. Efrat, S. Har-Peled, On finding a guard that sees most and a shop that sells most, in: Proc. 15th ACM-SIAM Symp. on Discrete Algorithms, 1098–1107, 2004.

[6] L.L. Deneen, S. Joshi, Treasures in an art gallery, in: Proc. 4th Canadian Conf. Computational Geometry, 17-22, 1992.

[7] S. Eidenbenz, (In-)Approximability of Visibility Problems on Polygons and Terrains, Ph.D. Thesis, ETH Zurich, 2000.

[8] S. Eidenbenz, Inapproximability Results for Guarding Polygons without Holes, in: Proc. ISAAC'98, Lecture notes in Computer Science, Vol. 1533 (Springer, Berlin, 1998) 427-436.

[9] S. Eidenbenz, C. Stamm, P. Widmayer, Inapproximability of some Art Gallery Problems, in: Proc. 10th Canadian Conf. Computational Geometry (CCCG'98), 64-65, 1998.
[10] I. Emiris, C. Fragoudakis, E. Markou, Maximizing the Guarded Interior of an Art Gallery, in: Proc. 22nd European Workshop on Computational Geometry, 165–168, 2006.
[11] S. Ghosh, Approximation algorithms for Art Gallery Problems, in: Proc. Canadian Information Processing Society Congress, 429-434, 1987.
[12] A. Laurentini, Guarding the walls of an art gallery, The Visual Computer Journal 15 (1999) 265-278.
[13] D. Lee, A. Lin, Computational complexity of art gallery problems, IEEE Trans. Inform. Theory 32 (1986) 276-282.
[14] E. Markou, Approximation algorithms for near - optimal covering of polygonal regions, Ph.D. Thesis, School of Electrical and Computer Engineering, National Technical University of Athens, 2003.
[15] E. Markou, C. Fragoudakis, S. Zachos, Approximating Visibility Problems within a constant, in: Proc. 3rd Workshop on Approximation and Randomization Algorithms in Communication Networks, 91-103, 2002.
[16] E. Markou, S. Zachos, C. Fragoudakis, Budgeted Coverage of a Maximum Part of a Polygonal Area, in: Proc. 1st Balkan Conference on Informatics, 174-182, 2003.
[17] B. J. Nilson, Guarding Art Galleries – Methods for Mobile Guards, Ph.D. Thesis, Lund University, 1995.
[18] B.J. Nilsson, Approximate Guarding of Monotone and Rectilinear Polygons, in: Proc. ICALP'05, 1362-1373, 2005.
[19] S. Ntafos, M. Tsoukalas, Optimum placement of guards, Info. Sciences 76 (1994) 141–150.
[20] J. O'Rourke, Art Gallery Theorems and Algorithms (Oxford Univ. Press, New York, 1987).
[21] T. Shermer, Recent results in Art Galleries, in: Proc. of the IEEE, 80: 1384-1399, 1992.
[22] J. Urrutia, Art gallery and Illumination Problems, in: J.-R. Sack and J. Urrutia, ed., Handbook of Computational Geometry, (North-Holland, Amsterdam, 2000) 973-1027.