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Gathering asynchronous oblivious mobile robots in a ring

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Abstract

We consider the problem of gathering identical, memoryless, mobile robots in one node of an anonymous unoriented ring. Robots start from different nodes of the ring. They operate in Look-Compute-Move cycles and have to end up in the same node. In one cycle, a robot takes a snapshot of the current configuration (Look), makes a decision to stay idle or to move to one of its adjacent nodes (Compute), and in the latter case makes an instantaneous move to this neighbor (Move). Cycles are performed asynchronously for each robot. For an odd number of robots we prove that gathering is feasible if and only if the initial configuration is not periodic, and we provide a gathering algorithm for any such configuration. For an even number of robots we decide feasibility of gathering except for one type of symmetric initial configurations, and provide gathering algorithms for initial configurations proved to be gatherable.

Keywords: asynchronous, mobile robot, gathering, ring.

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1 Introduction

Mobile entities (robots), initially situated at different locations, have to gather at the same location (not determined in advance) and remain in it. This problem of distributed self-organization of mobile entities is known in the literature as the gathering problem. The main difficulty of gathering is that robots have to break symmetry by agreeing on a common meeting location. This difficulty is aggravated when (as in our scenario) robots cannot communicate directly but have to make decisions about their moves only by observing the environment.

We study the gathering problem in a scenario which, while very simple to describe, makes the symmetry breaking component particularly hard. Consider an unoriented anonymous ring of stations (nodes). Neither nodes nor links of the ring have any labels. Initially, some nodes of the ring are occupied by robots and there is at most one robot in each node. The goal is to gather all robots in one node of the ring and stop. Robots operate in Look-Compute-Move cycles. In one cycle, a robot takes a snapshot of the current configuration (Look), then, based on the perceived configuration, makes a decision to stay idle or to move to one of its adjacent nodes (Compute), and in the latter case makes an instantaneous move to this neighbor (Move). Cycles are performed asynchronously for each robot. This means that the time between Look, Compute, and Move operations is finite but unbounded, and is decided by the adversary for each robot. The only constraint is that moves are instantaneous, and hence any robot performing a Look operation sees all other robots at nodes of the ring and not on edges, while performing a move. However a robot R may perform a Look operation at some time t, perceiving robots at some nodes, then Compute a target neighbor at some time $t' > t$, and Move to this neighbor at some later time $t'' > t'$ in which some robots are in different nodes from those previously perceived by R because in the meantime they performed their Move operations. Hence robots may move based on significantly outdated perceptions, which adds to the difficulty of achieving the goal of gathering. It should be stressed that robots are memoryless (oblivious), i.e., they do not have any memory of past observations. Thus the target node (which is either the current position of the robot or one of its neighbors) is decided by the robot during a Compute operation solely on the basis of the location of other robots perceived in the previous Look operation. Robots are anonymous and execute the same deterministic algorithm. They cannot leave any marks at visited nodes, nor send any messages to other robots.

This very weak scenario, similar to that considered in [1, 3, 5, 6, 10, 13, 14], is justified by the fact that robots may be very small, cheap and mass-produced devices. Adding distinct labels, memory, or communication capabilities makes production of such devices more difficult, and increases their size and price, which is not desirable. Thus it is interesting to consider such a scenario from the point of view of applications. On the theoretical side, this weak scenario increases the difficulty of gathering by making the problem of symmetry breaking particularly hard, and thus provides an interesting setting to study this latter issue in a distributed environment.

It should be noted that the gathering problem under the scenario described above is related to the well-known leader election problem (cf. e.g. [12]) but is harder than it for the following reason. If robots in the initial configuration cannot elect a leader among nodes (this happens for all periodic configurations and for some symmetric configurations) then gathering is impossible (see Section 3). However, even if leader election is possible in the initial configuration, this does not necessarily guarantee feasibility of gathering. Indeed, while the node elected as a leader is a natural candidate for the place to gather, it is not clear how to preserve the same target node during the gathering process, due to its asynchrony. (Recall that nodes do not have labels, and configurations perceived by robots during their Look operation change during the gathering process, thus robots may not "recognize" the previously elected node later on.)

An important and well studied capability in the literature on robot gathering is the *multiplicity detection* [10, 14]. This is the ability of the robots to perceive, during the Look operation, if there is one or more robots in a given location. In our case, we prove that without this capability, gathering of more than one robot is always impossible. Thus we assume the capability of multiplicity detection in our further considerations. It should be stressed that, during a Look operation, a robot can only tell if at some node there are no robots, there is one robot, or there are more than one robots: a robot does not see a difference between a node occupied by a or b robots, for distinct $a, b > 1$.

1.1 Related work

The problem of gathering mobile robots in one location has been extensively studied in the literature. Many variations of this task have been considered. Robots move either in a graph, cf. e.g. [2, 7, 8, 9, 11], or in the plane [1, 3, 4, 5, 6, 10, 13, 14, 15], they are labeled [7, 8, 11], or anonymous [1, 3, 4, 5, 6, 10, 13, 14, 15], gathering algorithms are probabilistic (cf. [2] and the literature cited there), or deterministic [1, 3, 4, 5, 6, 7, 9, 10, 11, 13, 14, 15]. Deterministic algorithms for gathering robots in a ring (which is a task closest to our current setting) have been studied e.g., in [7, 8, 9, 11]. In [7, 8, 11] symmetry was broken by assuming that robots have distinct labels, and in [9] it was broken by using tokens.

To the best of our knowledge, the very weak assumption of anonymous identical robots that cannot send any messages and communicate with the environment only by observing it, was used to study deterministic gathering only in the case of robots moving freely in the plane [1, 3, 4, 5, 6, 10, 13, 14, 15]. The scenario was further precised in various ways. In [4] it was assumed that robots have memory, while in [1, 3, 5, 6, 10, 13, 14, 15] robots were oblivious, i.e., it was assumed that they do not have any memory of past observations. Oblivious robots operate in Look-Compute-Move cycles, similar to those described in our scenario. The differences are in the amount of synchrony assumed in the execution of the cycles. In [3, 15] cycles were executed synchronously in rounds by all active robots, and the adversary could only decide which robots are active in a given cycle. In [4, 5, 6, 10, 13, 14, 15] they were executed asynchronously: the adversary could interleave operations arbitrarily, stop robots during the move, and schedule Look operations of some robots while others were moving. It was proved in [10] that gathering is possible in the asynchronous model if robots have the same orientation of the plane, even with limited visibility. Without orientation, the gathering problem was positively solved in [5], assuming that robots have the capability of multiplicity detection. A complementary negative result concerning the asynchronous model was proved in [14]: without multiplicity detection, gathering robots that do not have orientation is impossible.

Our scenario is the most similar to the asynchronous model used in [10, 14]. The only difference is in the execution of Move operations. This has been adapted to the context of the ring of stations (nodes): moves of the robots are executed instantaneously from a node to its neighbor, and hence robots always see other robots at nodes. All possibilities of the adversary concerning interleaving operations performed by various robots are the same as in the model from [10, 14], and the characteristics of the robots (anonymity, obliviousness, multiplicity detection) are also the same.

1.2 Our results

For an odd number of robots we prove that gathering is feasible if and only if the initial configuration is not periodic, and we provide a gathering algorithm for any such configuration. For an even number of robots we decide feasibility of gathering except for one type of symmetric configurations, and provide gathering algorithms for initial configurations proved to be gatherable.

2 Terminology and preliminaries

We consider an n-node anonymous unoriented ring. Initially, some nodes of the ring are occupied by robots and there is at most one robot in each node. The number of robots is denoted by k. During the gathering process robots move, and at any time they occupy some nodes of the ring, forming a *configuration*. A configuration is denoted by a pair of sequences $((a_1, \ldots, a_r), (b_1, \ldots, b_s))$, where the integers a_i and b_j have the following meaning. Choose an arbitrary node occupied by at least one robot as node u_1 and consider consecutive nodes $u_1, u_2, u_3, \ldots, u_r$, occupied by at least one robot, starting from u_1 in the clockwise direction. (Clockwise direction is introduced only for the purpose of definition, robots do not have this notion, as the ring is not oriented.) Integer a_i, for $i < r$, denotes the distance in the ring between nodes u_i and u_{i+1}, and integer a_r denotes the distance between nodes u_r and u_1 (in the clockwise direction). Next, consider those nodes among $u_1, u_2, u_3, \ldots, u_r$ which are occupied by more than one robot. Such nodes are called *multiplicities*. Suppose that $u_{v_1}, \ldots, u_{v_s}$ are these consecutive nodes (ordered in clockwise direction). Integer b_i is defined as the distance in the clockwise direction between node u_1 and node u_{v_i}. It should be clear that different choices of node u_1 give rise to different pairs of sequences. Respective sequences in these pairs are cyclic shifts of each other and correspond to the same positioning of robots. So formally a configuration should be defined as an equivalence class of a pair of sequences with respect to those shifts. To simplify notation we will use pairs of sequences instead of those classes, and for configurations without multiplicities we will drop the second sequence, simply using sequence $(a_1, \ldots, a_r)$. An example of a configuration with two multiplicities is shown in Figure 1.

Consider a configuration $C = (a_1, \ldots, a_r)$ without multiplicities. The *range* of the configuration C is the set $\{a_1, \ldots, a_r\}$. For any integer a_i in the range of C, the *weight* of a_i is the number

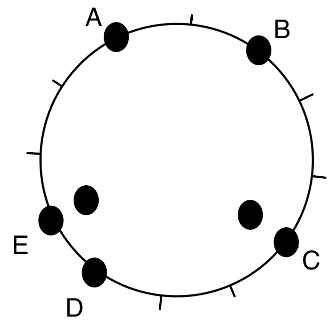

Figure 1: A configuration with two multiplicities. The pair of sequences describing this configuration starting from robot A is $((2,3,3,1,3), (5,9))$. The view of robot A is $\{((2,3,3,1,3),(5,9)),((3,1,3,3,2),(3,7))\}$.

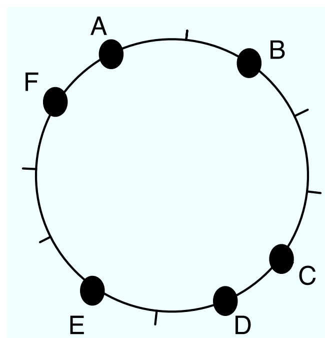

Figure 2: A periodic configuration. The sequence describing this configuration starting from robot A is $(2,3,1,2,3,1)$. The view of robots A and D is $\{(2,3,1,2,3,1),(1,3,2,1,3,2)\}$. Robots B and E have the same view $\{(3,1,2,3,1,2),(2,1,3,2,1,3)\}$. Robots C and F have the same view $\{(1,2,3,1,2,3),(3,2,1,3,2,1)\}$.

of times this integer appears in the sequence $(a_1,\ldots,a_r)$. C is called *periodic* if the sequence $(a_1,\ldots,a_r)$ is a concatenation of at least two copies of a subsequence p. A periodic configuration is shown in Figure 2. The configuration C can be also represented as the set Z of nodes occupied by the robots. C is called *symmetric* if there exists an axis of symmetry of the ring, such that the set Z is symmetric with respect to this axis. If the number of robots is odd and S is an axis of symmetry of the set Z then there is exactly one robot on the axis S. This robot is called *axial* for this axis. A symmetric configuration is shown in Figure 3. Notice that all cases are possible for a configuration: symmetric, periodic, both symmetric and periodic, neither symmetric nor periodic. A configuration which is periodic and symmetric is shown in Figure 4. Two robots are called *neighboring*, if at least one of the two segments of the ring between them does not contain any robots. A segment of the ring between two neighboring robots is called *free* if there is no robot in this segment.

We now describe formally what a robot perceives during a Look operation. Fix a robot R in

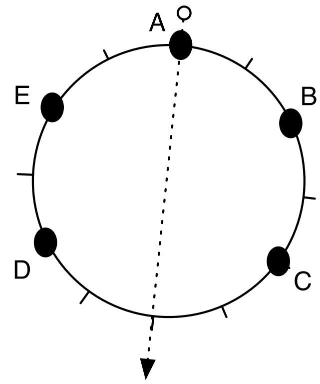

Figure 3: A symmetric configuration. The sequence describing this configuration starting from robot A is $(2,2,4,2,2)$. The view of robot A is $\{(2,2,4,2,2),(2,2,4,2,2)\}$. Robots B and E have the same view $\{(2,4,2,2,2),(2,2,2,4,2)\}$. Robots C and D have the same view $\{(4,2,2,2,2),(2,2,2,2,4)\}$.

a configuration represented by a pair of sequences $((a_1,\ldots,a_r),(b_1,\ldots,b_s))$, where this particular representation is taken with respect to the node occupied by R (i.e., this node is considered as node u_1). The *view* of robot R is the set of two pairs of sequences $\{((a_1,\ldots,a_r),(b_1,\ldots,b_s)),((a_r,a_{r-1},\ldots,a_1),(n-b_s,\ldots,n-b_1))\}$ (if the node occupied by R is a multiplicity then we define the view of R as $\{((a_1,\ldots,a_r),(0,b_2,\ldots,b_s)),((a_r,a_{r-1},\ldots,a_1),(0,n-b_s,\ldots,n-b_2))\}$). This formalization captures the fact that the ring is unoriented and hence the robot R cannot distinguish between a configuration and its symmetric image, if R is itself on the axis of symmetry. This is conveyed by defining the view as the *set* of the two couple of sequences because the sets

$$\{((a_1,\ldots,a_r),(b_1,\ldots,b_s)),((a_r,a_{r-1},\ldots,a_1),(n-b_s,\ldots,n-b_1))\}$$

and

$$\{((a_r,a_{r-1},\ldots,a_1),(n-b_s,\ldots,n-b_1)),((a_1,\ldots,a_r),(b_1,\ldots,b_s))\}$$

are equal. As before, if there are no multiplicities, we will drop the second sequence in each case and write the view as the set of two sequences: $\{(a_1,\ldots,a_r),(a_r,a_{r-1},\ldots,a_1)\}$. For example, in a 9-node ring with consecutive nodes $1,\ldots,9$ and three robots occupying nodes 1,2,4, the view of robot R at node 1 is the set $\{(1,2,6),(6,2,1)\}$.

A configuration without multiplicities is called *rigid* if the views of all robots are distinct. A rigid configuration is shown in Figure 5. We will use the following geometric facts.

Lemma 2.1 *1. A configuration without multiplicities is non-rigid, if and only if it is either periodic or symmetric.*

2. If a configuration without multiplicities is non-rigid and non-periodic then it has exactly one axis of symmetry.

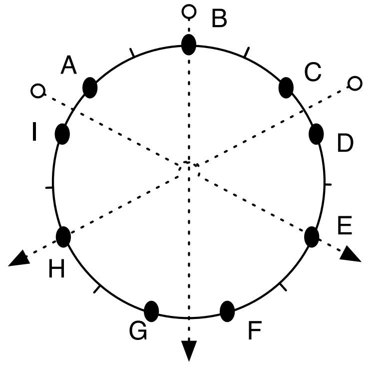

Figure 4: A symmetric and periodic configuration. The sequence describing this configuration starting from robot A is $(2,2,1,2,2,1,2,2,1)$. This configuration has 3 axes of symmetry. The view of robots A, C, D, F, G, I is $\{(2,2,1,2,2,1,2,2,1),(1,2,2,1,2,2,1,2,2)\}$. The view of robots B, E, H is $\{(2,1,2,2,1,2,2,1,2),(2,1,2,2,1,2,2,1,2)\}$.

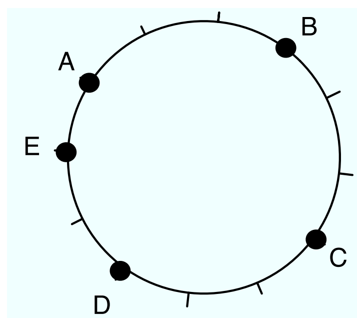

Figure 5: A rigid configuration. The views of robots A, B, C, D, E are $\{(3,3,3,2,1),(1,2,3,3,3)\}$, $\{(3,3,2,1,3),(3,1,2,3,3)\}$, $\{(3,2,1,3,3),(3,3,1,2,3)\}$, $\{(2,1,3,3,3),(3,3,3,1,2)\}$ and $\{(1,3,3,3,2),(2,3,3,3,1)\}$ respectively.

Proof:

1. For the first part of the Lemma, we first show that if a configuration C without multiplicities is symmetric then C is non-rigid. Consider two robots a, b (placed at symmetric nodes) with views $\{(a_1,\ldots,a_r),(a_r,a_{r-1},\ldots,a_1)\}$, $\{(b_1,\ldots,b_r),(b_r,b_{r-1},\ldots,b_1)\}$. Suppose that in a's view, the sequence of distances $a_+ = (a_1,\ldots,a_r)$ is in the clockwise direction and in b's view, the sequence of distances $b_- = (b_r,b_{r-1},\ldots,b_1)$ is in the counterclockwise direction (again clockwise and counterclockwise directions are introduced only for the purpose of analysis, robots do not have these notions, as the ring is not oriented). The axis of symmetry S crosses the ring in two points. Let a_s be the first and a_m be the second distance in a_+ which correspond to segments crossed by the axis of symmetry. The first intersection point is either at the middle of a_s or at the node u_{s+1} containing a robot c (recall from Section 2 that a_s denotes the distance in the ring between nodes u_s and u_{s+1}).

The second intersection point is either at the middle of a_m or at the node u_{m+1} containing a robot d. Let also a_w be the distance in a_+ for which u_{w+1} is the node where robot b has been placed. Then the $w-$th distance in b_- is the distance between nodes u_2 and u_1 (u_1 is the node where robot a has been placed). Because a and b are placed at symmetric nodes, $a_1 = b_r = a_w = b_{r-w+1}$ and the distances a_s, a_m appear as $s-$th, $m-$th, respectively in b_-. Also $a_r = b_1$ for the same reason. Moreover for every j, if the $j-$th distance in a_+ lies between any two of a_1, a_s, a_w, a_m, a_r then it appears as the $j-$th distance in b_- and lies between the corresponding two of $b_r, b_{r-s+1}, b_{r-w+1}, b_{r-m+1}, b_1$. Therefore:

$$(a_1, \dots, a_s, \dots, a_w, \dots, a_m, \dots, a_r) = (b_r, \dots, b_{r-s+1}, \dots, b_{r-w+1}, \dots, b_{r-m+1}, \dots, b_1).$$

This means that robots a, b have the same view and hence the configuration is non-rigid.

If the configuration C is periodic then it can be written as $C = (a_1, \dots, a_r)$ and C is a concatenation of at least two copies of a sequence $(a_1, \dots, a_p)$ of distances. It holds:

$$(a_1, \dots, a_p, \dots, a_r) = (a_{p+1}, \dots, a_r, a_1, \dots, a_p)$$

since the second sequence is only the first one shifted by the repeated sequence $(a_1, \dots, a_p)$. Let $a_+ = (a_1, \dots, a_p, \dots, a_r)$ be the sequence of distances in the clockwise direction in a robot a's view (robot a is placed at node u_1). Then $(a_{p+1}, \dots, a_r, a_1, \dots, a_p)$ is the sequence of distances in the clockwise direction in a robot b's view (where robot b is placed at node u_{p+1}). Therefore a, b have the same views and the configuration is again non-rigid.

Now we prove that if C is non-rigid then it is either symmetric or periodic. Since C is non-rigid, there are two robots a, b having the same view. Suppose again that in a's view, the sequence of distances $a_+ = (a_1, \dots, a_r)$ is in the clockwise direction and the sequence of distances $a_- = (a_r, a_{r-1}, \dots, a_1)$ is in the counterclockwise direction and analogous for b: $b_+ = (b_1, \dots, b_r)$ and $b_- = (b_r, b_{r-1}, \dots, b_1)$.

Thus either $(a_1, \dots, a_r) = (b_r, b_{r-1}, \dots, b_1)$, or $(a_1, \dots, a_r) = (b_1, \dots, b_r)$.

- If $(a_1, \dots, a_r) = (b_r, b_{r-1}, \dots, b_1)$: Let a_w be the distance in a_+ for which u_{w+1} is the node where robot b has been placed. Hence $a_w = b_r$. Then the $w-$th distance in b_- is the distance between nodes u_2 and u_1 (u_1 is the node where robot a has been placed). Hence $b_{r-w+1} = a_1$. Let a_s be the $\lfloor \frac{w+1}{2} \rfloor$-th distance in a_+ (between a_1 and a_w) and let a_m be the $\lfloor \frac{w+r}{2} \rfloor$-th distance in a_+ (between a_w and a_r). We have, $a_1 = b_r = a_w = b_{r-w+1}$ and the distances a_s, a_m appear as $s-$th, $m-$th, respectively in b_-. Also from the hypothesis it holds $a_r = b_1$. Moreover for every j, if the $j-$th distance in a_+ lies between any two of a_1, a_s, a_w, a_m, a_r then it appears as the $j-$th distance in b_- and lies between the corresponding two of $b_r, b_{r-s+1}, b_{r-w+1}, b_{r-m+1}, b_1$. Therefore:

  $$(a_1, \dots, a_s, \dots, a_w, \dots, a_m, \dots, a_r) = (b_r, \dots, b_{r-s+1}, \dots, b_{r-w+1}, \dots, b_{r-m+1}, \dots, b_1).$$

  The configuration is symmetric and the axis of symmetry crosses the distance a_s in the middle or at u_{s+1} and the distance a_m in the middle or at u_{m+1}.

- If $(a_1, \ldots, a_r) = (b_1, \ldots, b_r)$: Suppose also that going clockwise from robot a to robot b there is no other robot c for which $c_+ = a_+$ (if there is such a robot then take that robot as b). Consider the distance a_w in a_+ for which u_{w+1} is the node where robot b has been placed. It must hold $a_{w+1} = b_1 = a_1$, $a_{w+2} = b_2 = a_2$ and so on. Hence $a_{w+i} = b_i = a_i$, $1 \leq i \leq w$ and $a_{mw+i} = b_i = a_i$, where $mw + i < r$. We claim that the above sequences are periodic, and more precisely that each one of them is a concatenation of copies of the subsequence $(a_1, \ldots, a_w)$. If r is a multiple of w then it is clearly the case. Otherwise, let $r = mw + x$, where $1 \leq x < w$ (i.e., $a_r = a_{mw+x} = b_x = a_x$). Consider the distance $a_{r+w-x} = a_{w-x}$. We have $a_{r+w-x} = a_{(m+1)w} = b_{mw}$. Therefore $a_{r+w-x+1} = b_{mw+1} = b_1 = a_1$ and in general $a_{r+w-x+i} = b_{mw+i} = b_i = a_i$. But a_{w-x} appears before a_w and consequently a_{w-x+1} appears before $a_{w+1} = b_1$. This means that the robot c which is at the node u_{w-x+1} (which is between a and b in a_+ sequence) has the sequence c_+ in its view equal to a_+ which is a contradiction.

2. For the second part of the Lemma, consider a non-rigid and non-periodic configuration. In view of the first part of the Lemma, the configuration has to be symmetric with an axis of symmetry S_1. Suppose that there is another axis of symmetry S_2. Take two robots a, b which are placed at symmetric nodes with respect to S_1. Then, as we proved in the first part of the Lemma, if $a_+ = (a_1, \ldots, a_r)$ is the sequence of distances in the clockwise direction in a's view and $b_- = (b_r, b_{r-1}, \ldots, b_1)$ is the sequence of distances in the counterclockwise direction in b's view, we have $a_+ = b_-$. If at least one of these robots (say a) is on the axis S_2 then $(a_1, \ldots, a_r) = (a_r, a_{r-1}, \ldots, a_1)$ and hence $(a_1, \ldots, a_r) = (b_1, \ldots, b_r)$. But then we can argue as before that the configuration is periodic, which leads to a contradiction. If none of a, b is on the axis S_2 then take a third robot c which has the same view as b because of the axis S_2. This means that $(b_1, \ldots, b_r) = (c_r, c_{r-1}, \ldots, c_1)$ (where $c_- = (c_r, c_{r-1}, \ldots, c_1)$ is the sequence of distances in the counterclockwise direction in c's view). Hence $(a_1, \ldots, a_r) = (c_1, \ldots, c_r)$. Again we can argue as before that the configuration is periodic, which is a contradiction.

□

Consider a configuration without multiplicities that is non-rigid and non-periodic. Then it is symmetric. Let S be its unique axis of symmetry. If the number of robots is odd then exactly one robot is situated on S and S goes through the antipodal node if the size n of the ring is even, and through the (middle of the) antipodal edge if n is odd. If the number of robots is even then two cases are possible:

- *edge-edge symmetry* : S goes through (the middles of) two antipodal edges;
- *node-on-axis symmetry* : at least one node is on the axis of symmetry.

Note that the first case can occur only for an even number of robots in a ring of even size.

We now establish two basic impossibility results. Note that similar results have been proven for gathering robots on the plane. However, these results do not directly imply ours.

Proposition 2.1 *1. Gathering any 2 robots is impossible on any ring.*

2. If multiplicity detection is not available then gathering any $k > 1$ robots is impossible on any ring.

Proof:

1. Consider a gathering algorithm for 2 robots. In any configuration the robots have the same view. Consider what the algorithm does if the distance between the robots is 1. If the algorithm tells the robots not to move then it is clearly incorrect. If it tells them to move then a synchronous adversary that schedules all operations of both robots simultaneously does not permit gathering: the robots will always be at odd distance (in the case when the algorithm tells them to move towards each other when at distance 1, this adversary forces perpetual swapping).

2. The proof is by induction on k. For $k = 2$ it follows from part 1. Suppose that the statement is true for all numbers $k' < k$ of robots and consider a gathering algorithm for k robots. Consider the configuration C just before the first multiplicity is created. Then at least one robot R moves to an adjacent node occupied by another robot. Consider an adversary that first schedules a Look and a Move operation only for robot R, and only then schedules the next Look operations for other robots. Robot R will create a multiplicity, thus reducing the number of nodes occupied by robots to $k - 1$. All subsequent Look operations will be performed for at most $k-1$ nodes occupied by robots. Since multiplicity detection is not available, the perceptions of the robots will be the same as in the case of less than k robots. By the inductive hypothesis gathering is thus impossible.

□

Proposition 2.1 justifies the two assumptions made throughout this paper: the number k of robots is at least 3 and robots are capable of multiplicity detection.

All our algorithms describe the Compute part of the cycle of robots' activities. They are written from the point of view of a robot R that got a view in a Look operation and computes its next move on the basis of this view.

The rest of the paper is organized as follows. In Section 3 we establish two impossibility results: gathering is not feasible for periodic and edge-edge symmetric configurations. In Section 4 we give a procedure to gather configurations containing exactly one multiplicity. In Section 5 we propose a gathering procedure for rigid configurations. In Section 6 we give the complete solution of the gathering problem for any odd number of robots. Section 7 concludes the paper with a discussion of gathering for an even number of robots and with open problems.

3 Impossibility results

In this section we show two impossibility results. The first one concerns any number of robots.

Theorem 3.1 *Gathering is impossible for any periodic configuration.*

Proof: Consider a periodic configuration with the period repeated $t > 1$ times. Consider an adversary synchronously scheduling all operations in rounds: first the Look operation for all robots, then the Compute operation for all robots, then the Move operation for all robots, and so on. The configuration is periodic in round 0. Suppose it is periodic in round r. The views of all t corresponding robots in the t copies of the period are identical in round r and hence the configuration remains periodic in round $r + 1$, with t copies of the period. By induction, the configuration remains periodic in every round, with t copies of the period. Since $t > 1$, gathering will never occur. □

The second impossibility result concerns only the case of an even number of robots on a ring of even size.

Theorem 3.2 *Gathering is impossible for any edge-edge symmetric configuration.*

Proof: Consider a configuration which has an edge-edge symmetry. This means that both the size of the ring and the number of robots are even. Consider an adversary synchronously scheduling all operations in rounds: first the Look operation for all robots, then the Compute operation for all robots, then the Move operation for all robots, and so on. The configuration is symmetric in round 0. Suppose it is symmetric in round r. If robot R' is the symmetric image of robot R with respect to this symmetry then the distance between R and R' is odd. Robots R and R' have identical views in round r, hence they will behave identically, and their distance in round $r + 1$ will change either by 0, or by 2 or by -2. This implies that in round $r + 1$ the configuration will remain symmetric (robots R and R' have again the same view), and the distance between robots R and R' will remain odd. By induction, the configuration remains symmetric, and the distance between a robot and its symmetric image remains odd, in all rounds. This implies that gathering will never occur. □

4 Gathering configurations with a single multiplicity

In this section we show a gathering procedure for any configuration containing exactly one multiplicity, say at node v. The idea is to gather all robots at v, avoiding creating another multiplicity (which could potentially create a symmetry, making the gathering process harder or even impossible). The following procedure achieves this goal by first moving the robots closest to v towards v, then moving there the second closest robots, and so on.

Procedure Single-Multiplicity-Gathering

if R is at the multiplicity **then** do not move
else
    if none of the segments between R and the multiplicity is free
    then do not move
    else move towards the multiplicity along the shortest of the free
        segments or along any of them in the case of equality.

Lemma 4.1 *Procedure Single-Multiplicity-Gathering performs gathering of robots for any configuration with a single multiplicity.*

Proof: The procedure guarantees that a robot moves only in the case when some segment of the ring between it and the multiplicity is free, and in this case the robot moves on this free segment. This implies that no multiplicity other than the existing one will be created. Since for any configuration with a single multiplicity, some robot outside of the multiplicity has a free segment between itself and the multiplicity, after any point of time t some robot outside of the multiplicity will make a move towards the multiplicity, always in the same direction. This implies that, if at time t there are still robots outside of the multiplicity, then at some later time t', some robot will reach the multiplicity, thus reducing the number of robots outside it. Since robots at the multiplicity never move, gathering will be eventually performed. □

5 Gathering rigid configurations

In this section we show a gathering procedure for any rigid configuration, regardless of the number of robots. The main idea of the procedure is to elect a single robot and move it until it hits one of its neighboring robots, thus creating a single multiplicity, and then to apply Procedure Single-Multiplicity-Gathering. The elected robot must be such that during its walk the rigidity property is not lost. In order to achieve this goal, we perform the election as follows. First the robots elect a pair of neighboring robots at maximum distance (there may be several such pairs, whence the need for election). Then they choose among them the robot which has the other neighboring robot closer. Ties can be broken easily (see the details of the algorithm).

In order to elect a robot we need to linearly order all possible views. This can be done in many ways. One of them is to order lexicographically all finite sequences of integers and number them by consecutive natural numbers. Then a view becomes a set of two natural numbers. Treat these sets as ordered pairs of natural numbers in increasing order, order these pairs lexicographically, and assign them consecutive natural numbers in increasing order. We fix the resulting linear order of views and this numbering beforehand, adding it to the algorithm for all robots. The natural number assigned to a view will be called the *code* of this view.

Procedure Rigid-Gathering

Max := the largest of the distances a_i in the view of R.
M := the robot with the largest code of view having a neighboring robot at distance Max.
N := the robot with the largest code of view having M as a neighboring robot at distance Max.
//the pair of robots at distance Max is elected.//
$j := 1$
$M_1 := M$; $N_1 := N$; $M_0 := N$; $N_0 := M$
repeat
    $j := j + 1$
    M_j := the neighboring robot of M_{j-1} different from M_{j-2}
    N_j := the neighboring robot of N_{j-1} different from N_{j-2}
until (the distance between N and N_j is different than
    the distance between M and M_j)
$N' := N_j$; $M' := M_j$
if there is no multiplicity **then**
    if the distance between N and N' is smaller than
    the distance between M and M' **then**
        if $R = N$ **then** move towards N_2
    else if $R = M$ **then** move towards M_2
else Single-Multiplicity-Gathering

Lemma 5.1 *Procedure Rigid-Gathering performs gathering of robots for any rigid configuration without multiplicities.*

Proof: Suppose that robots M and N at distance Max are elected in the first part of the procedure. Suppose, without loss of generality, that the distance a between M and M' is less than the distance b between N and N'. Then robot M moves towards M_2. After this move, the distance between M and N becomes $Max + 1$, and the distance between M and M_2 becomes $a - 1$. No other distances between neighboring robots change. Hence in the new configuration, M and N are again neighboring robots at maximum distance. The configuration is again rigid because M and N are the unique pair of neighboring robots at distance $Max+1$ and the distance $a - 1$ between M and M_2 is smaller than the distance b between N and N_2. Robots M and N are again elected because now there is only one neighboring pair of robots at the maximum distance $Max + 1$. Since the distance $a - 1$ between M and M_2 is smaller than the distance b between N and N_2, it is again the robot M that will move towards M_2. It follows that, until a multiplicity is created, only one robot will move, and it will move in the same direction. This guarantees that a multiplicity will be finally created and it will be unique. Hence Procedure Single-Multiplicity-Gathering will be applied, thus completing gathering, in view of Lemma 4.1. □

6 Gathering an odd number of robots

In this section we present a gathering algorithm for any non-periodic configuration of an odd number of robots. Together with Theorem 3.1 this solves the gathering problem for an odd number of robots. The idea of the algorithm is the following. Consider any non-periodic configuration of an odd number of robots (recall that initially there are no multiplicities). If it is rigid then apply Procedure Rigid-Gathering. Otherwise it must be symmetric, by Lemma 2.1. There is a unique axial robot for its unique axis of symmetry. Move this robot to any adjacent node. We prove that three cases can occur. (1) The resulting situation has a multiplicity (the adjacent node was occupied by a robot): then apply Procedure Single-Multiplicity-Gathering. (2) The resulting configuration is rigid: then apply Procedure Rigid-Gathering. (3) Another axis of symmetry has been created (the previous one has been obviously destroyed by the move). In this case there is a unique axial robot for the unique axis of symmetry. Move this robot to any adjacent node. Again one of the three above cases can occur. We prove that after a finite number of such moves, only cases (1) or (2) can occur, and thus gathering is finally accomplished either by applying Procedure Single-Multiplicity-Gathering or by applying Procedure Rigid-Gathering.

Hence the algorithm can be stated as follows.

Algorithm Odd-Gathering

if the configuration is periodic **then** output: gathering impossible
else
    if the configuration has a single multiplicity
    then Single-Multiplicity-Gathering
    else
        if the configuration is rigid **then** Rigid-Gathering
        else
            if R is axial **then** move (to any of the adjacent nodes)

Example 6.1 *Consider the configuration* $C = (a, a+1, a+1, a+1, a+1, a+1, a)$ *of 7 robots, for some* $a > 1$*. This is a symmetric non-periodic configuration with the axial robot at distance* a *from its neighboring robots. After moving the axial robot towards one of its neighboring robots, we obtain the configuration* $C' = (a-1, a+1, a+1, a+1, a+1, a+1, a+1,)$*, which is again symmetric and non-periodic. Its axial robot is at distance* $a+1$ *from its neighboring robots. After moving the axial robot of* C' *towards one of its neighboring robots, we obtain the configuration* $C'' = (a+2, a+1, a+1, a-1, a+1, a+1, a)$*, which is rigid. Now gathering is completed using Procedure Rigid-Gathering.* □

In the proof of the correctness of Algorithm Odd-Gathering we will use the following lemmas.

Lemma 6.1 *Let C be a symmetric configuration of an odd number of robots, without multiplicities. Let C' be the configuration resulting from C by moving the axial robot to any of the adjacent nodes. Assume that C' does not have multiplicities. Then C' is not periodic.*

Proof: Since C is symmetric, C' is of the form $(a+1, b_1, \ldots, b_{s-1}, b_s, b_{s-1}, \ldots, b_1, a-1)$. Suppose that C' is periodic and take the period d of a length p in which $d_1 = a+1$ (the first term of the period d is the first term of C'). Then $d_p = a-1$ (the last term of the period d is the last term of C'). Since C is symmetric, it must hold that $d_p = d_1$. Contradiction. $\square$

Lemma 6.2 *Let C be a symmetric non-periodic configuration of an odd number of robots, without multiplicities. Then exactly one value in the range of C has odd weight.*

Proof: By Lemma 2.1, the configuration C has exactly one axis of symmetry. Let S be this axis, and let A and B be the unique pair of neighboring robots, situated on both sides of S. Let x be the length of the free segment between A and B. Consider any value y different from x, in the range of C. For every pair of neighboring robots with the free segment between them of length y, there is the symmetric pair of robots with the same length of the free segment between them. This implies that y must have even weight. On the other hand, for every pair of neighboring robots different from A, B, with the free segment between them of length x, there is the symmetric pair of robots with the same length of the free segment between them. In view of the existence of the pair A, B, this implies that x must have odd weight. $\square$

Let C be a symmetric non-periodic configuration of an odd number of robots, without multiplicities. The unique value of odd weight in the configuration C is called the *chief* of C. The distance between the axial robot and its neighboring robots is called the *index* of C. Let C' be the configuration resulting from C by moving the axial robot to any of the adjacent nodes. If C' does not have multiplicities and is symmetric then we will call it *special*.

Lemma 6.3 *Let C be a symmetric non-periodic configuration of an odd number of robots, without multiplicities. Let x be the index of C. Let C' be the configuration resulting from C by moving the axial robot to any of the adjacent nodes. If C' is special then the chief of C' is either $x+1$ or $x-1$.*

Proof: Let A be the axial robot of the configuration C, and let U and W be the two neighboring robots of A. Let S_1 be the unique axis of symmetry of C and S_2 the unique axis of symmetry of C'. For any pair of neighboring robots $\{R_1, R_2\}$ in configuration C', let $F_2(\{R_1, R_2\})$ be the symmetric image of the pair $\{R_1, R_2\}$ by the axis of symmetry S_2. For any pair of neighboring robots $\{R_1, R_2\}$ in configuration C', different from $\{A, U\}$ and from $\{A, W\}$, let $F_1(\{R_1, R_2\})$ be the symmetric image of the pair $\{R_1, R_2\}$ by the axis of symmetry S_1. (Note that S_1 is indeed an axis of symmetry for configuration C' if we delete robot A from it.) Consider the pair of robots $P = \{A, U\}$ and the sequence of pairs $(P, F_2(P), F_1(F_2(P)), F_2(F_1(F_2(P))), \ldots)$. Consider the first term T of this sequence equal to some previous term (it must exist because the number of pairs of robots is finite). Let T' be the term immediately preceding T. We cannot have $T = P$ for the following reason. If $T = F_2(T')$ then we would have $T' = F_2(P)$, contrary to the definition of T, and if $T = F_1(T')$, this would imply that robots A, U and A, W are equidistant in configuration C', which is false. We cannot have $T = F_1(T'')$, or $T = F_2(T'')$, for $T'' \neq T'$, because this would imply that T' is equal to some previous term, contrary to the definition of T. Hence the only remaining possibility is $T = T'$. If $T = F_1(T')$ then T is the unique pair of neighboring robots on both sides of S_1, and if $T = F_2(T')$ then T is the unique pair of neighboring robots on both sides of S_2. The same argument is valid for the pair $\{A, W\}$ instead of $\{A, U\}$. This shows that either the pair of neighboring robots situated on both sides of S_1 is at distance $x - 1$ and the pair of neighboring robots situated on both sides of S_2 is at distance $x + 1$, or vice-versa. Since S_2 is the axis of symmetry of configuration C', in the first case the chief of C' is $x + 1$ and in the second case it is $x - 1$. □

Consider a special configuration C. The subset of the range of C consisting of integers of the same parity as that of the chief is called the *white part* of the range, and its complement is called the *black part* of the range. We denote by $b(C)$ the total number of occurrences in C of integers from the black part of its range.

Lemma 6.4 *Consider a special configuration C with index z and chief f. Let C' be the configuration resulting from C by moving the axial robot to any of the adjacent nodes. If $z \neq f - 1$ and $z \neq f + 1$ then C' is not special.*

Proof: Suppose that C' is special. By assumption we have $f \neq z - 1$ and $f \neq z + 1$. Hence the parity of the weight of f does not change in C', and thus it remains odd. On the other hand, by Lemma 6.3, the chief of C' is either $z + 1$ or $z - 1$. Hence there are at least two values in the range of C' with odd weight, which contradicts Lemma 6.2. □

Lemma 6.5 *Consider a special configuration C. Let C' be the configuration resulting from C by moving the axial robot to any of the adjacent nodes. If C' is special then $b(C') < b(C)$.*

Proof: Let z be the index of C, and f its chief. By Lemma 6.4, we have either $z = f - 1$ or $z = f + 1$. Consider the first case, i.e., $f = z + 1$. In the configuration C', the weight of each of the integers $z + 1$ and $z - 1$ increases by 1 and the weight of integer z decreases by 2. Since the weight of f was odd in C, now it becomes even. Since the weight of $z - 1$ was even, now it becomes odd and $z - 1$ is the chief of C'. The parity of the chief does not change with respect to the configuration C. Hence z (if it still has positive weight in C') is in the black part of the range of C' because it has parity different from that of the chief. Integers $z - 1$ and $z + 1$ are in the white part of the range. No other weights change in comparison with C. It follows that the sum of weights in the black part of C' is by 2 smaller than the sum of weights in the black part of C. The argument in the second case, i.e., when $f = z - 1$, is analogous. □

Corollary 6.1 *Consider a sequence $(C_1, C_2, \ldots)$ of special configurations, such that C_{i+1} results from C_i by moving the axial robot to any of the adjacent nodes. Then for some $i \leq k$, we have $b(C_i) = 0$.*

Lemma 6.6 *Consider a special configuration C, with $b(C) = 0$. Let C' be the configuration resulting from C by moving the axial robot to any of the adjacent nodes. If C' does not have multiplicities then it is not symmetric.*

Proof: If C' were symmetric, then it would be special. This contradicts Lemma 6.5 in view of $b(C) = 0$. □

We are now ready to prove the correctness of Algorithm Odd-Gathering.

Theorem 6.1 *Algorithm Odd-Gathering performs gathering of any non-periodic configuration of an odd number of robots.*

Proof: Consider an initial non-periodic configuration C of an odd number of robots. By assumption it does not contain multiplicities. If it is rigid then we are done by Lemma 5.1. Otherwise, it must be symmetric by Lemma 2.1. Let A be its unique axial robot. Let C_1 be the configuration resulting from C by moving robot A to any of the adjacent nodes. If C_1 contains a multiplicity then we are done by Lemma 4.1. If C_1 is rigid then we are done by Lemma 5.1. Otherwise, C_1 is either periodic or symmetric, in view of Lemma 2.1. By Lemma 6.1, it cannot be periodic, hence it must be symmetric, and thus special. Consider the configuration C_2 resulting by moving the axial robot of C_1 to any of the adjacent nodes. Again C_2 either contains a multiplicity, or is rigid, or is special. In the first two cases we are done, and in the third case the axial robot is moved again. In this way we create a sequence $C_1, C_2, \ldots$ of special configurations. By Lemma 6.1, there is a configuration C_i in this sequence, with $b(C_i) = 0$. Let C' be the configuration resulting from C_i by moving the axial robot to any of the adjacent nodes. By Lemma 6.6, the configuration C' either has a multiplicity, or cannot be symmetric, and thus must be rigid. In the first case we are done by Lemma 4.1 and in the second case by Lemma 5.1. □

Theorem 6.1 and Theorem 3.1 imply the following corollary.

Corollary 6.2 *For an odd number of robots, gathering is feasible if and only if the initial configuration is not periodic.*

7 Conclusion

We completely solved the gathering problem for any odd number of robots, by characterizing configurations possible to gather (these are exactly non-periodic configurations) and providing a gathering algorithm for all these configurations. Corollary 6.2 is equivalent to the following statement: for an odd number of robots, gathering is feasible if and only if in the initial configuration, robots can elect a node occupied by a robot.

For an even number of robots, we proved that gathering is impossible if either the number of robots is 2, or the configuration is periodic, or when it has an edge-edge symmetry. On the other hand, we provided a gathering algorithm for all rigid configurations. This leaves unsettled one type of configurations: symmetric non-periodic configurations of an even number of robots with a node-on-axis type of symmetry. These are symmetric non-periodic configurations in which at least one node is situated on the unique axis of symmetry. This (these) node(s) may or may not be occupied by robots. In this case, the symmetry can be broken by initially electing one of the axial nodes. This node is a natural candidate for the place to gather. However, it is not clear how to preserve the same target node during the gathering process, due to its asynchrony. Unlike in our gathering algorithm for an odd number of robots, where only one robot moves until a multiplicity is created, in the case of the above symmetric configuration of an even number of robots, some robots would have to move together. This creates many possible outcomes of Look operations for other robots, in view of various possible behaviors of the adversary, which can interleave their actions. We note here that for an even number of robots there are cases where gathering is feasible even when robots cannot initially elect a node occupied by a robot.

The complete solution of the gathering problem for an even number of robots remains a challenging open question left by our research. We conjecture that in the unique case left open (non-periodic configurations of an even number of robots with a node-on-axis symmetry), gathering is always feasible. In view of our results, this is equivalent to the following statement.

Conjecture: For an even number of more than 2 robots, gathering is feasible if and only if the initial configuration is not periodic and does not have an edge-edge symmetry.

The validity of this conjecture would imply that, for any number of more than 2 robots, gathering is feasible if and only if, in the initial configuration robots can elect a node (not necessarily occupied by a robot).

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