Maximizing the Guarded Interior of an Art Gallery

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Abstract

In the Art Gallery problem a polygon is given and the goal is to place as few guards as possible so that the entire area of the polygon is covered. We address a closely related problem: how to place a fixed number of guards on the vertices or the edges of a simple polygon so that the total guarded area inside the polygon is maximized. We prove that the problem and some variations are APX-hard and we present polynomial time algorithms that achieve constant approximation ratios. Finally we extend our results for the case where the guards are required to cover valued items inside the polygon. The valued items or "treasures" are modeled as simple closed polygons.

1 Introduction

In the Art Gallery problem a polygon is given and the goal is to place as few guards as possible so that the entire area of the polygon is covered. Many variations of the Art Gallery problem have been studied during the last two decades ([11], [12], [13]). These can be classified with respect to where the guards are allowed to be placed (e.g. on vertices, edges, interior of the polygon) or whether only the boundary or all of the interior of the polygon needs to be guarded, etc. Most known variations are NP–hard.

We address a closely related problem:

Definition 1 *Given is a simple polygon P and an integer $k > 0$. The goal of the* Maximum Area Vertex Guards *problem is to place k vertex guards so that the area of P's interior that is overseen by the guards is maximum.*

The Minimum Vertex Guards problem asks how to guard a polygon, with or without holes, using a minimum number of guards placed on vertices; extensions consider edges or points in the interior. These problems are APX–hard and $O(\log n)$–approximable [8, 3, 4]. A related problem about terrain guarding, is the Minimum Fixed Height Vertex (Point) Guards On Terrain problem ($\Theta(\log n)$–approximable [6], [3], [4]). In [7] the case of guarding the walls (and not necessarily every interior point) is studied. In [2] the following problem has been introduced: suppose we have a number of valuable treasures in a polygon P; what is the minimum number of mobile (edge) guards required to patrol P in such a way that each treasure is always visible from at least one guard? In [2] they show NP–hardness and give heuristics for this problem. In [1] weights are assigned to the treasures in the gallery. They study the case of placing one guard in the gallery in such a way that the sum of weights of the visible treasures is maximized. Recent (non-)approximability results for art gallery problems can be found in [8, 11, 12, 13, 4, 6].

2 Finest Visibility Subdivision

We recall from [5] and [9] some preliminary definitions: Let P be a simple polygon, $a, b \in P$ two points inside P and $L, M \subseteq P$ two sets of points inside P. We say that point a *sees* point b, i.e. a and b are mutually visible, if the segment connecting a and b lies inside the closed polygon P. We say that the point set L is *visible* from the point set M or that M *oversees* L if for every point a which belongs to L, there exists a point b that belongs to M, such that a *sees* b. Notice that if M oversees L, it is not necessary for L to oversee M. Finally, M *watches* L if there exists a point a that belongs to L and a point b that belongs to M such that a *sees* b. Notice that if M watches L then also L watches M.

Our method descritizes the interior of any simple polygon with respect to visibility. In [5] we defined the notion of the *Finest Visibility Segmentation* of the boundary of a polygon P: Consider the visibility graph $V_G(P)$ with vertex set $V(P)$, i.e. the vertex set of P, where two vertices share an edge iff they are visible in P. By extending the edges of $V_G(P)$ inside P up to the boundary of P we obtain a set of points *FVS* of the boundary of P, that includes of course all vertices of P. An extended edge of $V_G(P)$ generates at most two *FVS* points. Since there are $O(n^2)$ edges in $V_G(P)$, there are $O(n^2)$ points in any polygon's *FVS* set. We call this construction the *Finest Visibility Segmentation* of the boundary of the polygon P. Any *open* segment (a, b), (i.e. a and b are excluded), defined by consecutive *FVS* points, is called an *FVS* segment of P.

Here we extend the *FVS* construction considering

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also all the intersection points of all the extended visibility graph's edges inside the polygon P. There are $O(n^4)$ regions created inside P which are called FVS regions of P. Due to the above construction, such an FVS region has the following property: none of the extended visibility edges can cross the region. We call this construction the *Finest Visibility Subdivision* of the interior of the polygon P.

The following two lemmas establish that a FVS region cannot be **only partly** visible from a vertex or an edge.

Lemma 1 *For any vertex v of a simple polygon P, a FVS region is visible by v if and only if it is watched by v.*

Proof. Of course if a FVS region is visible by v, then it is watched by v. Suppose now that the region is watched by v but not overseen by v. In that case there must be another vertex v' which blocks the visibility of v. But then vv' is a visibility edge and the extension of vv' crosses the FVS region which cannot hold since it contradicts the definition of a FVS region. □

Lemma 2 *For any edge e of a simple polygon P, a FVS region is visible by e if and only if it is watched by e.*

Proof. (**Sketch**) Of course if a FVS region is visible by e, then it is watched by e. Suppose now that the region is watched by $e = (v_i, v_j)$ but not overseen by e. This means that there is a point p inside the FVS region which sees a point $d \in e$. Sweeping the line that passes through p and d around d we meet a vertex v_m of P before the sweep lines stops crossing the FVS region (since the region is not overseen). We start now sweeping the line that passes through d and v_m around v_m until the line stops crossing the FVS region. Notice that if another vertex v_n (or v'_n) is hit by the sweep line then there must be a visibility edge (e.g. $v_m v_n$) whose extension crosses the FVS region. But the latter cannot hold since by definition an FVS region is not crossed by any visibility edges. □

The above lemmas demonstrate the following: The interior of any simple polygon P can be effectively descritized in terms of visibility to $O(n^4)$ FVS regions. Any vertex (edge) of P oversees a FVS region if and only if it watches the FVS region.

In order to find the set of all overseen FVS regions from a polygon vertex v, namely the $FVS(v)$ set, (using lemma 1) it suffices to select a point p inside every FVS region and connect it to vertex v. If this segment is everywhere inside the polygon P, then the region containing p is overseen from v.

For the case of the $FVS(e)$ set, that is the set of all overseen FVS segments from a polygon edge e, (using lemma 2) it suffices to select a point p inside every FVS region and then sweeping a line around p. The FVS region containing p is overseen by an edge touched by that line.

3 The Maximum Area Vertex (Edge) Guards problem

Let $A(r)$ be the area of the region r.

Algorithm 1 Maximum Area Vertex Guards

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compute the FVS regions
for all v ∈ V(P) do
  compute FVS(v)
end for
SOL ← ∅
for i = 1 to k do
  select v ∈ V that maximizes A(SOL ∪ FVS(v))
  SOL ← SOL ∪ FVS(v)
end for
return A(SOL)
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Consider P and integers $k, A > 0$. It is NP–hard to decide whether we can place at most k guards on vertices of P so that the total area overseen by the guards is at least A. To see why, consider the decision Minimum Vertex Guards problem, which asks whether the interior is overseen by at most k guards. The reduction to Maximum Area Vertex Guards is straightforward.

In [10] they prove that to maximize the guarded boundary of a polygon with or without holes using guards placed on vertices or edges, is APX–hard. If we change the construction part of the reduction so that to make sure that the area touched by the (so called) "cheap edges" is small enough then the following theorem holds:

Theorem 3 *The* Maximum Area Vertex (Edge) Guards *problem is APX–hard.*

Algorithm 1 is an approximation algorithm for the Maximum Area Vertex Guards problem. It starts by calculating the FVS regions and then for every $v \in V(P)$ the set $FVS(v)$. During each iteration of the algorithm, for any vertex v that hasn't been assigned a guard yet, the set $SOL \cup FVS(v)$ (of the overseen regions) is found and its area is calculated. The vertex that maximizes the total area of the (not previously) overseen regions is then chosen, causing a maximum possible increase of the solution. Then the algorithm updates the set SOL by adding the new FVS regions.

In order to prove that algorithm 1 approximates Maximum Area Vertex Guards by a constant approximation factor, we work as follows:

Let OPT denote the collection of the set of regions in an optimal solution and SOL denote the collection returned by the algorithm. These collections have $A(OPT)$ and $A(SOL)$ values respectively. Suppose that the algorithm places a guard at vertex v_i at iteration i, and a set of new regions P_i. Therefore the added total value of regions at iteration i is $A(P_i)$.

Consider the ordered sequence of vertices (as they have been selected by the algorithm) and let v_l be the first vertex in the sequence where a guard has been placed by the algorithm but not in the optimal solution. It holds:

$$A(P_i) = A(\cup_{m=1}^{i} P_m) - A(\cup_{m=1}^{i-1} P_m)$$

Lemma 4 *After l iterations of algorithm 1, we have, for $l = 1, 2, \ldots, k$,*

$$A(\cup_{i=1}^{l} P_i) - A(\cup_{i=1}^{l-1} P_i) \geq \frac{A(OPT) - A(\cup_{i=1}^{l-1} P_i)}{k}.$$

Proof. Consider vertices where guards have been placed in the optimal solution but no guard has been placed there by the algorithm. By the pigeonhole principle, there is at least one such vertex v_m so that the following holds:

$$A(P'_m) \geq \frac{A(OPT) - A(\cup_{i=1}^{l-1} P'_i)}{k}$$

We have:

$$A(P'_m) \geq \frac{A(OPT) - A(\cup_{i=1}^{l-1} P_i)}{k}$$

Notice that

$$A(P_l) \geq A(P_m) \geq A(P'_m)$$

and

$$A(\cup_{i=1}^{l} P_i) - A(\cup_{i=1}^{l-1} P_i) = A(P_l)$$

Therefore:

$$A(\cup_{i=1}^{l} P_i) - A(\cup_{i=1}^{l-1} P_i) \geq \frac{A(OPT) - A(\cup_{i=1}^{l-1} P_i)}{k}$$

□

Lemma 5 *After l iterations of algorithm 1 it holds:*

$$A(\cup_{i=1}^{l} P_i) \geq (1 - (1 - \frac{1}{k})^{l})A(OPT), \;\; l = 1, ..., k$$

Proof. We are going to prove this by induction on l. During the first step of the algorithm the set with value $A(P_1)$ is chosen. It holds:

$$A(P_1) \geq A(P'_1)$$

$A(P'_1)$ is the maximum possible value that OPT achieves, so from the pigeonhole principle:

$$A(P'_1) \geq \frac{A(OPT)}{k} \rightarrow A(P_1) \geq \frac{A(OPT)}{k}$$

Suppose that the relation holds for $i = l - 1$:

$$A(\cup_{i=1}^{l-1} P_i) \geq (1 - (1 - \frac{1}{k})^{l-1})A(OPT)$$

Since:

$$A(\cup_{i=1}^{l} P_i) = A(\cup_{i=1}^{l-1} P_i) + (A(\cup_{i=1}^{l} P_i) - A(\cup_{i=1}^{l-1} P_i))$$

using lemma 4 we have:

$$A(\cup_{i=1}^{l} P_i) \geq A(\cup_{i=1}^{l-1} P_i) + \frac{A(OPT) - A(\cup_{i=1}^{l-1} P_i)}{k} \rightarrow$$

$$A(\cup_{i=1}^{l} P_i) \geq A(\cup_{i=1}^{l-1} P_i)(1 - \frac{1}{k}) + \frac{A(OPT)}{k}$$

From the inductive hypothesis:

$$A(\cup_{i=1}^{l} P_i) \geq (1-(1-\frac{1}{k})^{l-1})A(OPT)(1-\frac{1}{k}) + \frac{A(OPT)}{k}$$

$$\rightarrow A(\cup_{i=1}^{l} P_i) \geq (1 - (1 - \frac{1}{k})^{l})A(OPT)$$

□

Theorem 6 *Algorithm 1 runs in polynomial time and achieves an approximation ratio of $\frac{1}{1-\frac{1}{e}} \approxeq 1.58$ for the* Maximum Area Vertex Guards *problem.*

Proof. Using lemma 5, we set $l = k$ and get:

$$A(\cup_{i=1}^{k} P_i) \geq (1 - (1 - \frac{1}{k})^{k})A(OPT)$$

It holds:

$$\lim_{k \rightarrow \infty} (1 - (1 - \frac{1}{k})^{k}) = 1 - \frac{1}{e}$$

As $(1 - (1 - \frac{1}{k})^{k})$ continuously gets smaller, we have:

$$1 - (1 - \frac{1}{k})^{k} \geq 1 - \frac{1}{e}$$

So:

$$A(SOL) > (1 - \frac{1}{e})W(OPT)$$

That is the algorithm approximates the Maximum Area Vertex Guards problem with a $\frac{1}{1-\frac{1}{e}} \rightarrow 1.58$ ratio. □

The algorithm's complexity is $O(n^4)$ because this is the size of FVS. A different approach, based on a triangulation and avoiding the FVS, yields $O(n^2k^2)$ for the same algorithm.

Similarly as before the Maximum Area Edge Guards problem is APX–hard. A similar algorithm to algorithm 1 approximates the Maximum Area Edge Guards problem. The only difference from algorithm 1 is that we need to calculate the $FVS(e)$ set using the technique described in section 2.

4 The Maximum Treasures Value Vertex (Edge) Guards problem

Consider the following problem: There exists a polygon which encloses a number of closed simple subpolygons each of which has a value assigned. The Maximum Treasures Value Vertex (Edge) Guards problem's goal is to place vertex or edge guards in a way which maximizes the total value of the overseen or watched subpolygons.

Proposition 7 *Given is a polygon P which encloses a number of closed simple subpolygons each of which has a value assigned. Given also are two integers $k, M > 0$. It is NP–hard to decide whether we can place at most k vertex or edge guards so that the total value of the overseen or watched subpolygons is at least M.*

Proof. (**Sketch**) The decision version of Minimum Vertex Guards problem for a polygon P reduces to the corresponding decision version of the Maximum Treasures Value Vertex (Edge) Guards problem for the same polygon: If M is the total number of the polygon's FVS regions then by assigning value 1 to each region the reduction is staightforward: The polygon's interior is overseen by at most k guards if and only if the total value of the overseen FVS regions is at least M. □

Theorem 8 Maximum Treasures Value Vertex (Edge) Guards *is APX–hard.*

Proof. (**Sketch**) We can change the construction part of the reduction presented in [10] by adding small enough subpolygons touching the "cheap edges". We assign a suitable small value to each subpolygon. □

Algorithm 1, with the appropriate modifications, approximates also the Maximum Treasures Value Vertex (Edge) Guards problem with the same ratio as in theorem 6. In fact, the computation of the FVS regions is not required for the case of vertex watching guards, since we can easily compute the watched subpolygons from any vertex. However, for the case of edge guards, a subpolygon p_i is watched by an edge e if and only if there is a FVS region watched by e that touches p_i. The total value of subpolygons in $A(SOL \cup FVS(v))$ is also required.

Notice that all the algorithms can be applied even when the polygons have holes.

5 Open problems

Interesting problems are the following: (a) How to place guards and given subpolygons in the polygon so that a maximum value is guarded (i.e. this time we need to place also the subpolygons). (b) How to place guards in the interior of P for all the above variations.

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